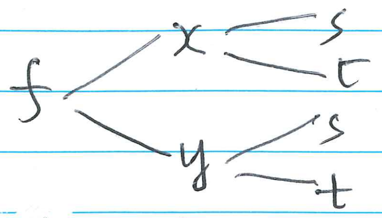
Excursus on Pf. of Chain Rule

Ex Suppose $f(x,y)$ is s.t. $x = x(s,t)$, $y = y(s,t)$, then

$$f = f(x(s,t), y(s,t)) = f(s,t)$$



Tree Diagram for f

Claim: $\dfrac{\partial f}{\partial t} = \dfrac{\partial f}{\partial x}\cdot\dfrac{\partial x}{\partial t} + \dfrac{\partial f}{\partial y}\cdot\dfrac{\partial y}{\partial t}$

Pf: $\dfrac{\partial f}{\partial t} = \lim\limits_{h\to 0} \dfrac{f(s,t+h) - f(s,t)}{h}$

$$= \lim_{h\to 0} \frac{f(x(s,t+h), y(s,t+h)) - f(x(s,t), y(s,t))}{h}$$

But the difference quotient could be rewritten as

$$\frac{f(x(s,t+h), y(s,t+h)) - f(x(s,t), y(s,t+h))}{h} \Leftarrow \mathrm{I}$$

$$+ \frac{f(x(s,t), y(s,t+h)) - f(x(s,t), y(s,t))}{h} \Leftarrow \mathrm{II}$$

Now that, we could rewrite I as

$$\mathrm{I} = \left[\frac{f(x(s,t+h), y(s,t+h)) - f(x(s,t), y(s,t+h))}{x(s,t+h) - x(s,t)}\right]\left[\frac{x(s,t+h) - x(s,t)}{h}\right]$$

$$= \frac{\partial f}{\partial x}(x^*, y(s,t+h))\left[\frac{x(s,t+h) - x(s,t)}{h}\right]$$

where $x(s,t) \leq x^* \leq x(s,t+h)$ (Mean Value Th^{m}.)

As $h \to 0$, $x^* \to x(s,t)$, $y(s,t+h) \to y(s,t)$

$\dfrac{x(s,t+h) - x(s,t)}{h} \longrightarrow \dfrac{\partial x}{\partial t}(s,t)$ by continuous differentiability

$\Rightarrow \mathrm{I} \longrightarrow \dfrac{\partial f}{\partial x}(x(s,t), y(s,t))\dfrac{\partial x(s,t)}{\partial t}$ Similarly for II.